Activity Report
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by

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Introduction. We have been working on the following issues:

1. Numerical methods and computational experiments for the multi-dimensional Inverse Scattering Problem (ISP) for the Helmholtz equation [3].
2. Numerical methods for the 3-dimensional ISP in *random* media [6].
3. *Global* convergence of numerical methods for 3-D ISP with time dependent data [7].
4. Phaseless 1-dimensional ISP [8].
5. Quasi-Reversibility method for nonclassical Cauchy problems [9]-[11].

Overall almost all our approaches are *entirely* new. In particular, we note that our numerical methods, for multidimensional ISPs see [3], [6], [7] and [11] work *directly* with *PDEs* rather than with corresponding *integral* equations. And this is a crucial aspect in which our methods differ from most others. This report is supplemented with our reprints/preprints [3], [5], [6] [8]-[12]. For this reason, we discuss all the issues rather briefly.

At the end of the report we briefly discuss applications of our results to ONR ARI SEA ICE INITIATIVE [2]. *We strongly believe that our research is very well suited for this initiative.*

1. Numerical Methods and Computational Experiments for Multidimensional ISP for the Helmholtz Equation [3].

In [13] we have described a version of quasi-Newton method for this ISP and presented results of numerical experiments. We have noted, however, that the presence of the highly oscillatory function $e^{ik<x,\nu>}$ in Helmholtz equation negatively affects our computational results in terms of both computer time and memory. Here the unit vector ν represents the direction of the incident plane wave.

Recently we have modified our method. Namely, let $u(x, \nu)$ be a solution of the forward scattering problem. We introduce the function $w(x, \nu) = e^{-ik<x,\nu>} u(x, \nu)$. The reader can compare $\mathrm{Re}[u(x, \nu)]$ and $\mathrm{Re}[w(x, \nu)]$ on figures 1 and 2, in the case of the scattering from a circular cylinder. For an additional discussion see [3]. Now we can use a *smaller* number of Fourier coefficients of the function w to obtain the same precision as before. As a result, our modified computational procedure requires *10 times less* computer time than before, in 2-D case. For instance, now the reconstruction process requires 8 sec. of CPU time instead of 84 sec., see [3].

Based on these results, we hope to conduct computational experiments in the 3-Dimensional case in the near future. Clearly such results are important for ARI [2].

2. 3-Dimensional ISP in Random Media [6]

We have noted in [6] that Forward Scattering Problems (ISP) in the random media can be successfully described by nonstationary Schrödinger equation (also see [17]). To our best knowledge, this equation has never been used by mathematicians to study multidimensional *Inverse Scattering Problems* either in random or in deterministic cases. Physicists accept a certain "deterministic" model for this random FSP, see [6], [17]. They formally derive this model from the random case. In this model, ensemble average and dispersion of the random density are engaged as real and imaginary parts of the potential accordingly. We have developed a quasi-Newton method for "deterministic" ISPs, which is naturally linked with the above mentioned "deterministic" FSP. We have established the convergence rate of our method. This ISP has significant applications in biology, underwater acoustic, and other areas.

3. Global Convergence of Numerical Method for 3-Dimensional ISP with Time Dependent Data [7]

We consider the following potential ISP for a hyperbolic equation. *Let* ϵ, T *be positive numbers and* $\epsilon < T$. *Suppose that function* $a(x) \in C^1(R^3)$. *Let this function be known for* $x \in \{|x| < \epsilon\}$. *Determine the funtion* $a(x)$ *for* $x \in \{|x| < \frac{1}{2}T\}$ *assuming that*

$$u_{tt} = \Delta u + a(x)u, \ (x, t) \in R^3 \times (0, T),$$

$$u|_{t=0} = 0, \ u_t|_{t=0} = \delta(x)$$

$$u|_{|x|=\epsilon} = p(x, t),$$

where the function $p(x, t)$ *is known.*

In [11] we have developed quasi-Newton method for similar ISP assuming that potential a(x) is sufficiently small. Recently we obtained a *fundamentally* new result. Namely, we introduce a special weighted minimized cost functional linked with Carleman estimates see [1], [12]. Then using these estimates we prove that our functional is ***strictly convex*** on each convex compact set of each finite dimensional space (by certain choice of parameters of Carleman estimate). Thus this result *guarantees* convergence of any version of the gradient method, if the starting point is chosen appropriately. W. Symes has obtained a similar result for the one-dimensional problem in [18]. He has also raised the question about the 3-D case.

4. Phaseless 1-Dimensional ISP [8]

In the classical formulation of 1-D ISP for Schrödinger equation, one has to determine the potential from the reflection coefficient R-(k). In many important applications, however, only the amplitude |R-(k)| can be measured. This ISP is tightly connected with the well known ***phase problem in optics*** see [4], [14], and [15]. **For the first time**, we have proved a uniqueness theorem and developed rigorous numerical method for this phaseless ISP.

5. Quasi-Reversibility Method and Stability for Nonclassical Cauchy Problems [5], [9]-[11].

It is commonly known that numerical methods for the nonclassical Cauchy problems are important value for numerical methods for inverse problems. The Quasi-Reversibility (QR) method was developed in [16]. But convergence rates were not established, and numerical experiments were conducted without noise in the data. Besides, the QR method has not been derived for the hyperbolic timelike Cauchy problem. The Carleman estimates play a fundamental role in the QR theory. We have shown this in [5], [9]-[11]. Using the Carleman estimates we established convergence rates for different versions of the QR method. Numerical experiments were conducted both in the case of the Laplace equation and the wave equation with noise in the data. These experiments demonstrate high efficiency and stability of the QR method.

6. Applications to ONR Sea Ice Initiative

Carefully reading [2] we have concluded that our research is closely related to challenges of the ARI. In order to demonstrate that, we draw some excerpts from [2] below and show how our results could work for these challenges.

1. [2, p. 2] *"Sea ice is inhomogeneous and highy variable at all spatial scales from millimeters to kilometers . . ."*

We think it would be reasonable to assume that small-scale variations of sea ice are random and large-scale variations are rather deterministic. Then if one would try to determine ensemble averages and dispersions of corresponding random fields, then our results [6] will work. Besides, we hope to extend results [6] on Maxwell systems. Further, numerical method [6] is based on the same idea as ones in [3]. Therefore computational experiments conducted in [3] could help to make computations for ISPs in *random* media. Besides, we hope to conduct numerical experiments in the 3-D case in the near future, and this case is definitely more realistic than 2-D cases.

2. [2, p.3] *"Can correlation functions . . . be successfully computed for modelling purposes . . ."*

The result of [6] is possibly related to this issue. In particular, this result can be applied to biological media, see also section 2 above. All our results concerning 3-D ISPs are closely related to several aspects metioned in the other questions in Section D of the ARI.

3. [2, p.3], bottom and top of p. 4

"There is need for the development and validation of time-dependent models for beam spread in sea ice . . . Mathematical models for EM inverse scattering must be developed to address these needs."

Our results in [7] and [11] respond positively to these challenges. The method in [7] is of special value since it is a global method, i.e., a good first guess is not required. Furthermore we hope to extend our methods to the more difficult and useful equation $u_{tt} = a^2(x)\Delta u$.

4. [2, p.7] *". . . have little theoretical guidance as to which measurements are most essential and the measurement accuracy required to draw strong conclusions"*

All our results together with uniqueness and stability theorems [1], [9] - [12] can provide a strong theoretical basis for this challenge. Besides, since numerical experiments show high stability of our methods, see [3], [9] and [10], they could help to predict the accuracy required. Furthermore, there are a large number of measurements for which only the amplitude of the signal is available, but not the phase [4]. Thus the phase *problem in optics* arises very naturally in these situations. Therefore our results [8], [14], and [15] can certainly be of use in such cases.

REFERENCES

1. A.L. Bukhgeim and M.V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, Soviet Math. Dokl. 24 (1981), 244-247.

2. Electromagnetic Properties of Sea Ice, results of an ONR Workshop, ONR, 1991.

3. S. Gutman and M.V. Klibanov, Two versions of quasi-Newton method for 3-dimensional inverse scattering problem, preprint, 1992.

4. N.E. Hurt, Phase Retrieval and Zero Crossings, Kluwer, Dordrecht, 1989.

5. M. Kazemi and M.V. Klibanov, Stability estimates for ill posed Cauchy problems involving hyperbolic equations and inequalities, Applicable Analysis, in print.

6. M.V. Klibanov, S. Gutman, and J. Malinsky, One approach to three-dimensional inverse scattering problem in random media, preprint, 1992.

7. M.V. Klibanov, Strict convexity of minimized cost functional for 3-dimensional hyperbolic inverse problem, in preparation.

8. M.V. Klibanov and P. Sacks, Phaseless inverse scattering and the phase problem in optics, submitted for publication, 1992.

9. M.V. Klibanov and Rakesh, Numerical solution of a timelike Cauchy problem for the wave equation, *Math. Methods in Appl Sciences*, in print.

10. M.V. Klibanov and F. Santosa, A computational quasi-reversibility method for Cauchy problems for Laplace's equation, *SIAM J. Appl. Math.*, 51 (1991), 1653-1675.

11. M.V. Klibanov and J. Malinsky, Newton-Kantorovich method for 3-dimensional potential inverse scattering problem and stability of the hyperbolic Cauchy problem with the time dependent data, *Inverse Problems*, 7 (1991), 577-596

12. M.V. Klibanov, Inverse problems and Carleman estimates, Inverse Problems, 8 (1992), August issue.

13. M.V. Klibanov and S. Gutman, Mathematical algorithms for multi-dimensional inverse scattering problems in inhomgeneous medium, ONR proposal, 1991.

14. M.V. Klibanov, On uniqueness of the determination of a compactly supported function from the modulus of its Fourier transform, *Soviet Math. Dokl.* 32 (1985), 668-670.

15. M.V. Klibanov, Determination of a function with compact support from the absolute value of its Fourier transform, and an inverse problem in optics, *Differential Equations*, 22 (1986), 1232-1240.

16. R. Lattes and J.-L. Lions, The Method of Quasi-Reversibility: Applications to Partial Differential Equations, Elsevier, New York, 1969.

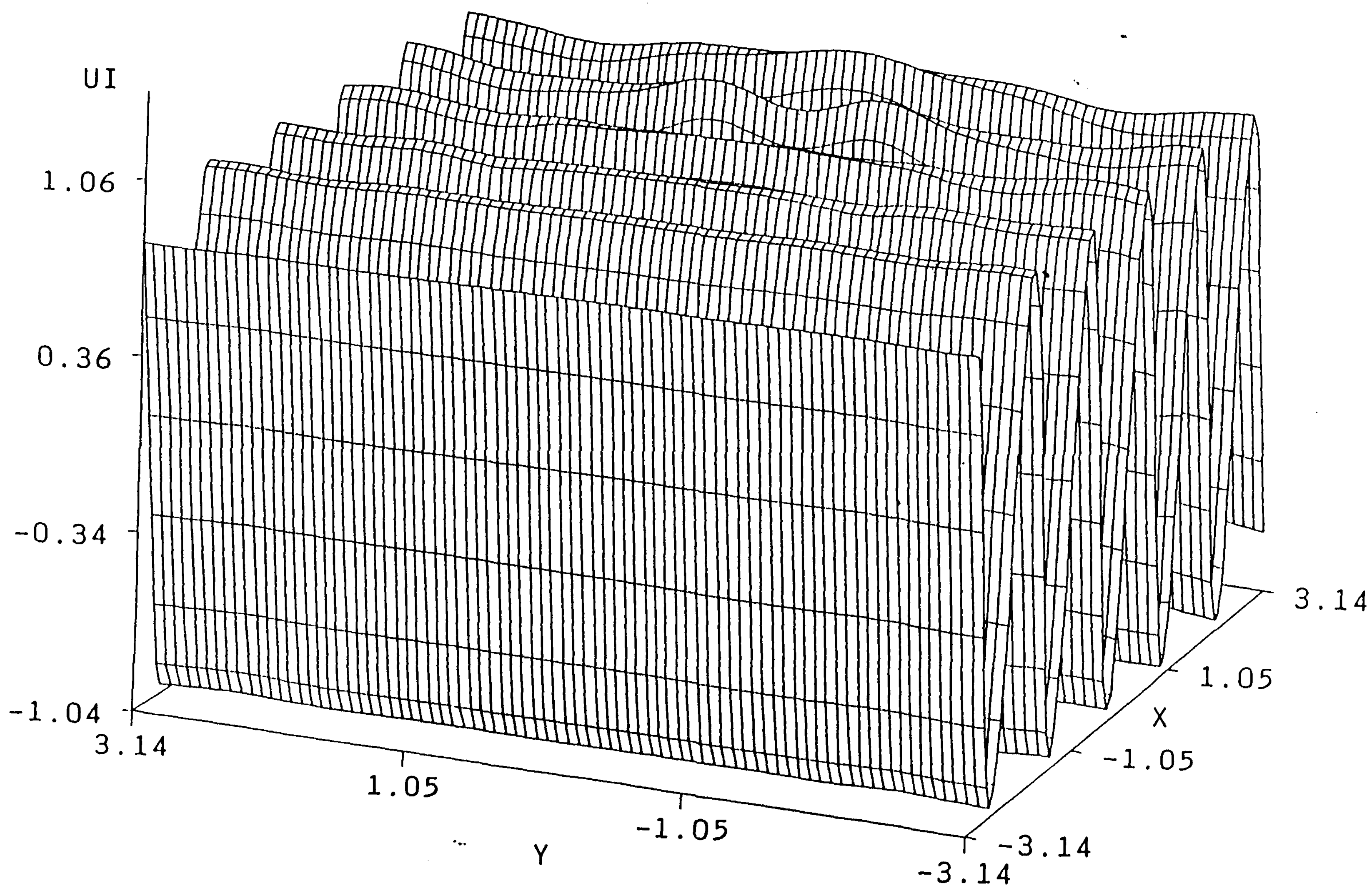
17. S.M. Rytov, Y.A. Kravtsov, and V.I. Tatarskii, Principles of Statistical Radiophysics, V. 2, Springer Verlag, New York, 1988, see Chapter 7.

18. W. Symes, Layered velocity inversion: A model problem from reflection seismology, SIAM J. Math. Anal., 22 (1991), 680-716.

Figure Captions

1. $\mathrm{Re}[u(x, \nu)]$, where $\nu = (1, 0)$, and u is scattering field from a circular cylinder with the radius $a = \frac{\pi}{2}$, $\epsilon = 0.03$ inside of the cylinder, $\epsilon = 0$ outside of the cylinder and $k = 5.3$. Here ϵ is a conductivity, see [3].
2. $\mathrm{Re}[u(x, \nu)e^{-ik<x,\nu>}]$ with the same coefficient ϵ as on the Figure 1. The features of the field are much more clear than ones on Figure 1.

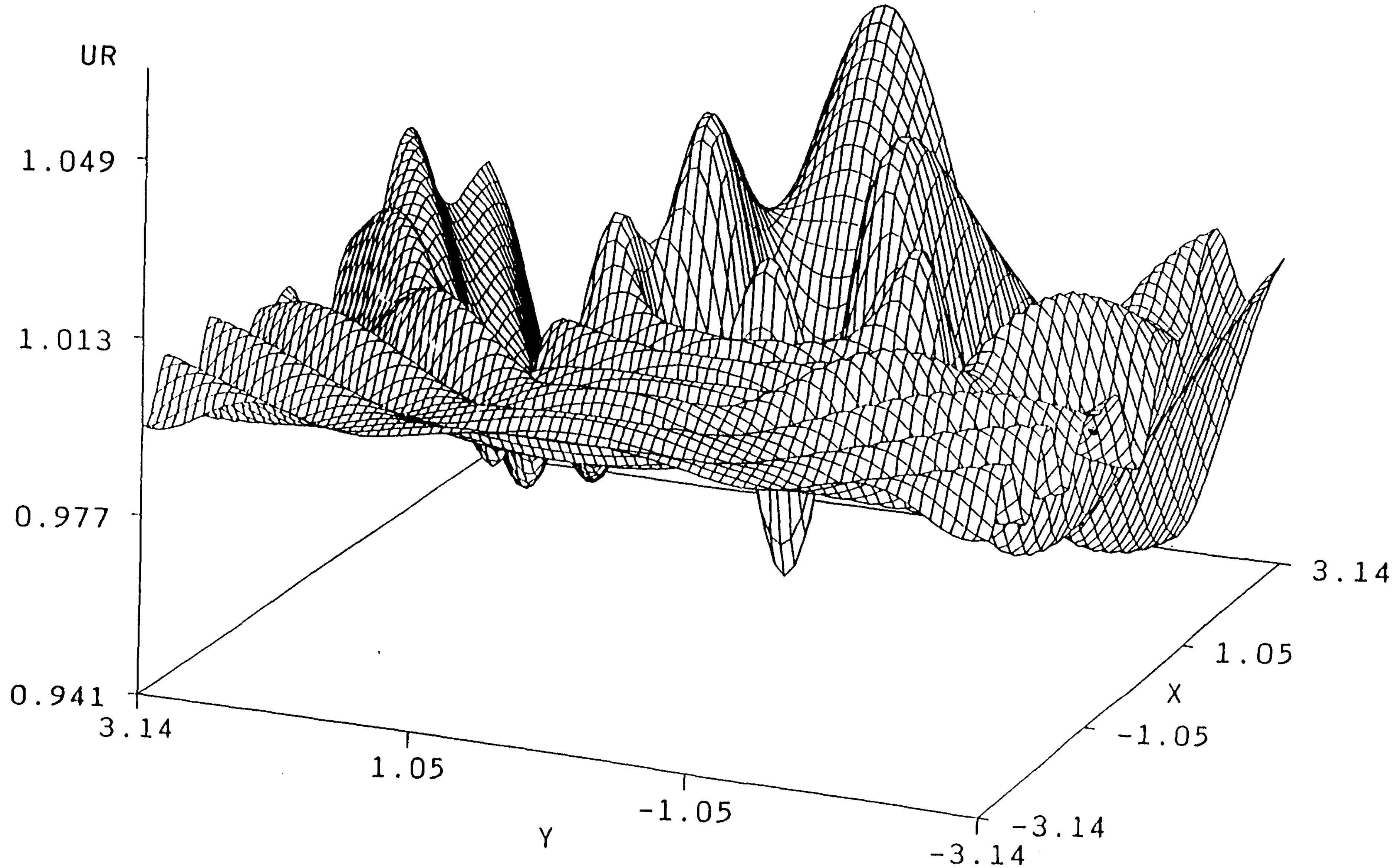
$Re[u(x,\nu)]$, $\nu = (1,0)$



k = 5.3 m = 7 a = 1.571 α = 0.03 b = 0.628 β = 0.03

Fig. 1

$\mathrm{RE}\left[u(x,\nu)\, e^{-ik\langle x,\nu\rangle}\right], \quad \nu = (1,0)$



$k = 5.3 \quad m = 7 \quad a = 1.571 \quad \alpha = 0.03 \quad b = 0.628 \quad \beta = 0.03$

Fig. 2